

ON THE LOWER GARLAND OF CERTAIN SUBGROUP LATTICES IN LINEAR GROUPS

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ABSTRACT. This paper is a revised and corrected version of [BP]. We describe here the lower garland of some lattices of intermediate subgroups in linear groups. The results are applied to the case of subgroup lattices in general and special linear groups over a class of rings, containing the group of rational points T of a maximal non-split torus in the corresponding algebraic group. It turns out that these garlands coincide with the interval of the whole lattice, consisting of subgroups between T and its normalizer. Bibliography: 10 titles.

INTRODUCTION

Let S be an associative ring with unit and R its unitary subring contained in the center of S . One can consider the following lattice of intermediate subgroups:

$$Lat(\text{Aut}({}_S S), \text{Aut}({}_R S)) = \{H : \text{Aut}({}_S S) \leq H \leq \text{Aut}({}_R S)\}$$

(see [BKH]). If S is a free left R -module of rank n , this lattice is isomorphic to the lattice of matrix subgroups

$$Lat(T, GL(n, R)) = \{H : T \leq H \leq GL(n, R)\},$$

where by $T = T(S)$ we denote the image of S^* under the inclusion which carries each element $\alpha \in S^*$ to the matrix of the operator of right multiplication by this element with respect to the fixed basis of this ring extension.

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Let R be commutative and $S = R^n$. Then we can consider R as a subring of S via “diagonal” embedding. Thus we get the lattice of subgroups in $\text{GL}(n, R)$, containing the group of diagonal matrices $\text{D}(n, R)$. The problem of description of this lattice has been solved for a rather wide class of rings (see the surveys [V1], [V2] for the background information).

If R is a field and S its finite extension, the achieved progress in the description of the lattice of intermediate subgroups is not so notable. The cases, when k is the field of real numbers, finite or local fields are investigated (qualitative or quantitative) completely or almost completely. The other known results are concerned with the quadratic extensions (see also [V1], [V2]).

At the same time a more general problem can be stated. Let $G' \leq \text{Aut}(R S)$. Then one can consider the subgroup lattice

$$\text{Lat}(\text{Aut}(S S) \cap G', G') = \{H' : \text{Aut}(S S) \cap G' \leq H' \leq G'\}$$

Of course, one can hardly expect to obtain complete description of this lattice for an arbitrary G' , but the case of finite field extensions and classical groups G' seems to be more acceptable.

The present paper is devoted to the investigation of certain properties of the group $\text{Aut}(S S) \cap G'$. Namely, several conditions on rings and groups which allow to calculate its normalizer in G' are proposed. Under some additional restrictions the lower garland of the lattice under consideration is described. These results are applied to the case of general and special linear groups.

Some results of this paper concerning the normalizer of $\text{Aut}(S S) \cap G'$ already appeared in [BP], but they were extremely incomplete in the form presented there (even the main “conjecture” of [BP] was trivial). Taking a chance, the author would like to ask to be excused by a potential reader of [BP].

Hereafter by a ring we mean an associative ring with unit. All subrings are assumed to be unitary and all modules to be left ones.

§ 1. THE NOTION OF A GARLAND

Let G be a group and G_0 its subgroup. One can be interested in the description of the lattice of intermediate subgroups

$$\text{Lat}(G_0, G) = \{H : G_0 \leq H \leq G\}$$

of G which contain G_0 .

Z.I.Borewicz [BKH] proposed a natural partition of this lattice into pairwise disjoint pieces which allows to reduce the primary problem to the problem of description of each of these pieces separately.

Namely, a structure of graph called the *normality graph* is introduced on the set $\text{Lat}(G_0, G)$. We take the intermediate subgroups H as the vertices of this graph

and link two of them with an edge iff one of the corresponding subgroups is normal in the other.

Definition. A *garland* of the lattice $Lat(G_0, G)$ is a connected component of the normality graph.

By this definition the lattice $Lat(G_0, G)$ is the disjoint set-theoretical union of its garlands. Thus, to describe the whole $Lat(G_0, G)$, it is sufficient to enumerate all the garlands and then to investigate each garland separately.

Subgroups G_0 and G are among intermediate, hence they belong to certain garlands. The garland containing G_0 is called *lower* and containing G *upper* garland of the lattice $Lat(G_0, G)$.

One can find a number of examples of decomposition of subgroup lattices in the union of their garlands in [AH].

§ 2. GENERAL CASE

We shall use the following notations. Let G be a group, $G_0 \leq G$. If $G' \leq G$ (an arbitrary subgroup), then $G'_0 = G_0 \cap G'$.

Let S be a ring and R its subring contained in the center of S . Let M be an S -module. We denote $G = \text{Aut}(R M)$, $G_0 = \text{Aut}(S M)$. Let $G' \leq G$.

We omit the proofs of all assertions of this section since they can be obtained in standard way.

Lemma 2.1. *The following assertions are equivalent:*

- (i) $\mathcal{N}_{G'} G'_0 \subseteq \mathcal{N}_G G_0 \cap G'$;
- (ii) if $G'_0 \trianglelefteq H' \leq G'$, then $H' \leq \mathcal{N}_G G_0$.

We shall assume that to the end of this section one of these two equivalent conditions is fulfilled.

Corollary 1. $\mathcal{N}_{G'} G'_0 = \mathcal{N}_G G_0 \cap G'$.

Let's denote by L_0 the lattice of subgroups $Lat(G_0, \mathcal{N}_G G_0)$, and by L'_0 the lattice $Lat(G'_0, \mathcal{N}_{G'} G'_0)$.

Corollary 2. $L'_0 = L_0 \cap G'$.

Let now S be a commutative ring, $M = S$. Then the group $G_0 \approx S^*$ is Abelian.

Corollary 3. *If G_0 is a maximal Abelian subgroup in G (e.g. if S is additively generated by its invertible elements), then G'_0 is also a maximal Abelian subgroup in G' .*

§ 3. CALCULATION OF THE NORMALIZER

Note that $\text{End}({}_S M)$ is an R -module since R is contained in the center of S . We assume that the following property holds true:

$$G'_0 \text{ additively generates (over } R) \text{ the group } G_0 \quad (+)$$

Lemma 3.1. $\mathcal{N}_{G'} G'_0 = \mathcal{N}_G G_0 \cap G'$.

Proof. Let $h \in \mathcal{N}_{G'} G'_0$, $t \in G_0$. Then $t = \sum_{i=1}^s \alpha_i t_i$, where $\alpha_i \in R$, $t_i \in G'_0$. Thus $h^{-1} t h = \sum_{i=1}^s \alpha_i h^{-1} t_i h = \sum_{i=1}^s \alpha_i t'_i$ for some $t'_i \in G'_0$. Hence $h^{-1} t h \in G_0$. The opposite inclusion is trivial.

Theorem 3.2. Let S be a ring and R its subring contained in the center of S . Assume that S is additively generated by its invertible elements. Let M be a free S -module of finite rank, G' be a subgroup of G such that the condition (+) is fulfilled. Then the normalizer of G'_0 in G' is equal to the intersection of the semidirect product of the normal subgroup G_0 and the subgroup isomorphic to the group $\text{Aut}(S/R)$ of all ring automorphisms of S , identical on R , with the group G' .

Proof. This theorem was proved in the paper of V.A.Koibaev [K] for $G' = G$. Therefore it is sufficient to apply Lemma 3.1.

We are primarily interested in the case $M = S$ (see Introduction). But the examination of the condition (+) even in this case is not possible for a sufficiently large class of rings. Moreover, this condition does not allow to obtain further information on the structure of the lower garland of the lattice $\text{Lat}(G'_0, G')$. So we shall try to impose another restriction on the class of rings under consideration.

Let S be a ring and R its subring, which is an integral domain contained in the center of S . Let also S be a free R -module of finite rank with a basis $\omega_1, \dots, \omega_n$.

We consider an embedding of S into the ring of matrices $M(n, R)$: each element $\alpha = \alpha_1 \omega_1 + \dots + \alpha_n \omega_n \in S$ (where $\alpha_i \in R$) maps to a matrix $t(\alpha) = (t_{ij}(\alpha))$, where $\omega_i \alpha = \sum_{j=1}^n t_{ji}(\alpha) \omega_j$.

Let's denote $T = t(S^*)$. Let k be the quotient field of R and \bar{k} its algebraic closure. We spread the mapping t to an affine space \mathbb{A}^n , changing R to \bar{k} and S to $\bar{k} \otimes_R S$. It follows from the definition of the mapping t that $t(\alpha\beta) = t(\beta)t(\alpha)$ for every $\alpha, \beta \in \bar{k} \otimes_R S$. We put $\tilde{T} = t((\bar{k} \otimes_R S)^*)$. It's clear that \tilde{T} is a group.

Lemma 3.3. \tilde{T} is a connected k -defined algebraic group.

Proof. It's clear that $\tilde{T} = t(U)$, where U is a Zariski open subset of \mathbb{A}^n . Therefore \tilde{T} is a connected algebraic group.

The set U is not changed after passing to S^{op} , and we get a morphism $t|_U : U \rightarrow \text{GL}(n, \bar{k})$ of k -groups, defined over k . Then its image is a k -subgroup of $\text{GL}(n, \bar{k})$.

Let G' be an arbitrary subgroup of $\mathrm{GL}(n, R)$. We put $T' = T \cap G'$. Assuming that \bar{k} is naturally embedded in $\bar{k} \otimes_R S$, we impose the following restriction on G' :

$$\bar{T} \text{ is additively generated over } \bar{k} \text{ by the connected component of } \bar{T}' \quad (*)$$

Lemma 3.4. $\mathcal{N}_{G'}T' = \mathcal{N}_{\mathrm{GL}(n, R)}T \cap G'$.

Proof. Let $g \in \mathcal{N}_{G'}T'$, i.e. $gT'g^{-1} = T'$. Since the conjugation is an isomorphism of algebraic groups, we see that $g\bar{T}'g^{-1} = \bar{T}'$. By the condition $(*)$ $gTg^{-1} \subseteq g\bar{T}g^{-1} \subseteq \tilde{T}$. Passing to “ R -points”, we get $g \in \mathcal{N}_{\mathrm{GL}(n, R)}T$.

Remark. In the course of the proof of Lemma 3.4 we used only the fact that \bar{T} is additively generated over \bar{k} by the whole group \bar{T}' .

Now we can formulate the analogue of Theorem 3.2.

Theorem 3.5. *Let S be a ring and R its subring, which is an integral domain contained in the center of S . Let S be a free R -module of finite rank which is additively generated by its invertible elements. Let G' be a subgroup of $\mathrm{GL}(n, R)$ such that the condition $(*)$ is fulfilled. Then the normalizer of T' in G' is equal to the intersection of the semidirect product of the normal subgroup T and the group $\mathrm{Aut}(S/R)$ of all ring automorphisms of S , identical on R , with the group G' .*

§ 4. CALCULATION OF THE LOWER GARLAND

Let, as above, S be a ring and R its subring, which is an integral domain contained in the center of S . We assume that S is a free R -module of rank n .

Let $G' = \tilde{G}(R)$, where \tilde{G} is a closed subgroup of $\mathrm{GL}(n, \bar{k})$.

Theorem 4.1. *Suppose that*

- (i) \tilde{G} is a connected reductive group;
- (ii) $\tilde{T} \cap \tilde{G}$ is a maximal torus in \tilde{G} ;
- (iii) the group T' is Zariski dense in $\tilde{T} \cap \tilde{G}$.

Then the lower garland of the lattice $\mathrm{Lat}(T', G')$ coincides with the interval $\mathrm{Lat}(T', \mathcal{N}_{G'}T')$.

Proof. One has to check that $T' \leq H \leq \mathcal{N}_{G'}T'$ implies $\mathcal{N}_{G'}H \leq \mathcal{N}_{G'}T'$.

Let $g \in \mathcal{N}_{G'}H$. Then $gT'g^{-1} \subseteq H \subseteq \mathcal{N}_{G'}T'$. Since the conjugation is an isomorphism of algebraic groups, we obtain $\mathcal{N}_{G'}T' \subseteq \mathcal{N}_{\tilde{G}}\bar{T}'$. The latter group is closed, hence $g\bar{T}'g^{-1} \subseteq \mathcal{N}_{\tilde{G}}\bar{T}'$. Since T' is dense in $\tilde{T} \cap \tilde{G}$, we get $g(\tilde{T} \cap \tilde{G})g^{-1} \subseteq \mathcal{N}_{\tilde{G}}(\tilde{T} \cap \tilde{G})$. It follows from (ii) that the (regular) torus $\tilde{T} \cap \tilde{G}$ coincides with its centralizer in \tilde{G} . But then the connected component $(\mathcal{N}_{\tilde{G}}(\tilde{T} \cap \tilde{G}))^o$ is equal to $\tilde{T} \cap \tilde{G}$. Since any morphism of algebraic groups maps the connected component of the first group onto the connected component of the second, we see that $g \in \mathcal{N}_{\tilde{G}}(\tilde{T} \cap \tilde{G})$. Passing to the “ R -points”, we get $g \in \mathcal{N}_{G'}T'$.

Note that in the course of the proof of Theorem 4.1 we did not use the explicit form of the normalizer of T' . Instead of this we imposed several restrictions on the groups, particularly, we assumed $\tilde{T} \cap \tilde{G}$ to be a maximal torus in \tilde{G} . Now we prove an analogue of Theorem 4.1 under some other conditions, and this allows us to get some corollaries which can not be deduced from Theorem 4.1.

Theorem 4.2. *Assume that S is additively generated by its invertible elements and the group $\text{Aut}(S/R)$ consisting of the ring automorphisms of S , which are identical on R , is finite. If the condition $(*)$ for G' is fulfilled, then the lower garland of the lattice $\text{Lat}(T', G')$ coincides with the interval $\text{Lat}(T', \mathcal{N}_{G'} T')$.*

Proof. The crucial idea in the proof of Theorem 4.1 lies in the fact that the connected component of the normalizer of $\overline{T'}$ coincides with $\overline{T'}$. Now the group $\overline{T'}$ may be not connected, but the scheme of the proof is the same.

Let $g \in \mathcal{N}_{G'} H$. It follows from Theorem 3.5 that $\mathcal{N}_{G'} T' = \bigcup_{i=1}^s \overset{\varnothing}{T' A_i}$ for some $A_i \in \text{GL}(n, R)$ and $A_1 = 1$. Thus $g \overline{T'} g^{-1} \subseteq \bigcup_{i=1}^s \overline{T' A_i}$. The latter set is a group, moreover it's a (closed) subgroup of $\text{GL}(n, \bar{k})$ and the index of its subgroup $(\overline{T'})^o$ is finite, hence this subgroup coincides with its connected component. Thus, as it was mentioned above, $g(\overline{T'})^o g^{-1} \subseteq (\overline{T'})^o$. Using $(*)$, we get $g T' g^{-1} \subseteq \tilde{T} \cap \tilde{G}$. Passing to the “ R -points”, we have $g \in \mathcal{N}_{G'} T'$.

Corollary. *In the settings of Theorem 4.1 or Theorem 4.2 $\mathcal{N}_{G'} \mathcal{N}_{G'} T' = \mathcal{N}_{G'} T'$.*

Remark. Usually these two theorems do not give any information on the structure of the lower garland in the case of noncommutative ring S . Indeed, then $\tilde{T} \cap \tilde{G}$ is not a torus, and the group $\text{Aut}(S/R)$ contains a lot of inner automorphisms, hence is not finite.

§ 5. SEPARABLE ALGEBRAS

Here we state without proofs some useful results on separable algebras. See [CR] for a more detailed exposition.

Let k be a field. All k -algebras assumed to be finite-dimensional.

Definition. A k -algebra A is called separable (over k), if the K -algebra $K \otimes_k A$ is semisimple for any extension K of the field k . In particular, any separable algebra is automatically semisimple.

Lemma 5.1. *A k -algebra A is separable iff the \bar{k} -algebra $\bar{k} \otimes_k A$ is semisimple.*

Theorem 5.2. *Let $A = A_1 \oplus \dots \oplus A_s$ be a decomposition of a semisimple k -algebra A into the direct sum of simple algebras $\{A_i\}$ with the centers $\{C_i\}$. Then A is separable iff all C_i are separable extensions of k .*

§ 6. THE CASE OF THE GENERAL LINEAR GROUP

Lemma 6.1. *Let S be a commutative ring and R its subring, which is an integral domain. Assume that S is a free R -module of rank n . If the \bar{k} -algebra $\bar{k} \otimes_R S$ is semisimple, then \tilde{T} is a maximal torus in $\mathrm{GL}(n, \bar{k})$ defined over k .*

Proof. In view of Lemma 3.3 and dimension arguments it remains to show that the group \tilde{T} is diagonalizable.

It follows from the settings that $\bar{k} \otimes_R S$ is isomorphic to the direct sum of n copies of the field \bar{k} . Thus \tilde{T} is conjugated to the group of diagonal matrices $D(n, \bar{k})$.

Remark. If the \bar{k} -algebra $\bar{k} \otimes_R S$ contains nilpotents, then \tilde{T} has unipotent elements, hence it is not a torus.

Theorem 6.2. *Let S be a commutative ring and R its subring, which is an integral domain. Assume that S is a free R -module of rank n , the \bar{k} -algebra $\bar{k} \otimes_R S$ is semisimple (where k is the quotient field of R), and T is Zariski dense in \tilde{T} . Then the lower garland of the lattice $\mathrm{Lat}(T, \mathrm{GL}(n, R))$ coincides with the interval $\mathrm{Lat}(T, \mathcal{N}_{\mathrm{GL}(n, R)} T)$.*

Proof. Follows from Theorem 4.1.

Remark. One can use also Theorem 4.2: it is sufficient to require that S is additively generated by its invertible elements, and the k -algebra $k \otimes_R S$ is semisimple, but then \tilde{T} may be not a torus.

Corollary. *Let $R = k$ be a field, $S = K_1 \oplus \dots \oplus K_t$, where K_i/k are finite extensions of k . Then:*

(i) *if all K_i/k are separable, then \tilde{T} is a maximal torus in $\mathrm{GL}(n, \bar{k})$ defined over k ;*

(ii) *if k is infinite, then the lower garland of the lattice $\mathrm{Lat}(T, \mathrm{GL}(n, k))$ coincides with the interval $\mathrm{Lat}(T, \mathcal{N}_{\mathrm{GL}(n, k)} T)$.*

Proof. Follows from Lemma 6.1 and results of § 5.

The case of a finite field will be treated in § 8.

Note that this Corollary generalizes the results of Al Hamad [AH].

Example. Let $R = \mathbb{Z}$ be the ring of integer numbers. Consider the ring $S = \mathbb{Z} \oplus \mathbb{Z}w$, where $w^2 = 0$. It's clear that there are only two automorphisms of S/R , namely, they map w to $\pm w$. Furthermore, the invertible elements of S are of the form $\pm 1 + bw$, where $b \in R$. Thus the first two conditions of Theorem 4.2 are fulfilled.

The group T is equal (if we choose elements $1, w$ as a basis of S/R) to the set of matrices

$$\left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a = \pm 1, b \in R \right\}$$

To get \overline{T} , one has to take an arbitrary element of $\overline{\mathbb{Q}}$ in place of b in the latter expression. We note that this group is not connected, but the condition $(*)$ is nevertheless fulfilled. Thus we can apply Theorem 4.2.

Note that it is impossible to apply Theorem 4.1 in this situation because of the existence of the unipotent elements in T .

§ 7. THE CASE OF THE SPECIAL LINEAR GROUP

Let $\tilde{G} = \mathrm{SL}(n, \overline{k})$.

Lemma 7.1. *Let S be a commutative ring and R its subring which is an integral domain. Let S be a free R -module of rank n . If the \overline{k} -algebra $\overline{k} \otimes_R S$ is semisimple, then $\tilde{T} \cap \tilde{G}$ is a maximal k -defined torus in $\mathrm{SL}(n, \overline{k})$.*

Proof. We show that the group $\tilde{T} \cap \tilde{G}$ is connected.

Since $\overline{k} \otimes_R S$ is semisimple, we see that $\tilde{T} \cap \tilde{G}$ is conjugated with the subgroup of $D(n, \overline{k}) \cap \tilde{G}$ consisting of diagonal matrices with the determinant 1. The latter group is connected since a polynomial $x_1 x_2 \dots x_n - 1$ is irreducible over \overline{k} .

To complete the proof it is sufficient to turn to the dimension arguments.

Theorem 7.2. *Let S be a ring and R its subring which is an integral domain containing in the center of S . Assume that S is a free R -module of rank n which is additively generated by its invertible elements, and T' is Zariski dense in $\tilde{T} \cap \tilde{G}$. Then the normalizer of the subgroup T' in G' is equal to the intersection of the semidirect product of the normal subgroup T and the group $\mathrm{Aut}(S/R)$ of the ring automorphisms of S , identical on R , with G' .*

Proof. It is clear that each matrix of \tilde{T} can be transformed to an element of $\tilde{T} \cap \tilde{G}$ via the multiplication by a scalar matrix, therefore the condition $(*)$ from § 3 is verified and we can apply Theorem 3.5 (see the remark after Lemma 3.4).

Theorem 7.3. *Let S be a commutative ring and R its subring which is an integral domain. Assume that S is a free R -module of rank n , the \overline{k} -algebra $\overline{k} \otimes_R S$ (where k is the quotient field of R) is semisimple, and T' is Zariski dense in $\tilde{T} \cap \tilde{G}$. Then the lower garland of the lattice $\mathrm{Lat}(T', \mathrm{SL}(n, R))$ is equal to the interval $\mathrm{Lat}(T', \mathcal{N}_{\mathrm{SL}(n, R)} T')$.*

Corollary. *Let $R = k$ be a field, $S = K_1 \oplus \dots \oplus K_t$, where K_i/k are finite separable extensions of k . Then:*

- (i) $\tilde{T} \cap \tilde{G}$ is a maximal k -defined torus in $\mathrm{SL}(n, \overline{k})$;
- (ii) if k is infinite, then the lower garland of the lattice $\mathrm{Lat}(T', \mathrm{SL}(n, k))$ is equal to the interval $\mathrm{Lat}(T, \mathcal{N}_{\mathrm{SL}(n, k)} T')$, which also equals the intersection of the lower garland of the lattice $\mathrm{Lat}(T, \mathrm{GL}(n, k))$ with $\mathrm{SL}(n, k)$.

The case of a finite field will be treated in § 8.

Now we analyse several examples.

1⁰. If K/k is a finite field extension, then $T' = \text{Ker } N_{K/k}$.

2⁰. Let $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{d})$ be its quadratic extension, where d is a squarefree integer rational number. If we take $1, \sqrt{d}$ as a basis of K/k , then

$$T' = \left\{ \begin{pmatrix} x & yd \\ y & x \end{pmatrix} : x, y \in \mathbb{Q}, x^2 - dy^2 = 1 \right\}$$

It is clear that there are only two automorphisms of S/R , namely, identical one and the linear mapping which carries \sqrt{d} to $-\sqrt{d}$.

Now we can explicitly calculate the subgroup $\mathcal{N}_{\text{SL}(2, \mathbb{Q})} T'$. Indeed, it follows from the Corollary of Theorem 7.3 that the result depends on the arithmetical properties of d (namely, on the solvability in rational numbers of the equation $x^2 - dy^2 = -1$): if d is positive and do not have any prime divisors of the form $4m + 3$, then

$$\begin{aligned} \mathcal{N}_{\text{SL}(2, \mathbb{Q})} T' &= T' \cup \left\{ \begin{pmatrix} x & -yd \\ y & -x \end{pmatrix} : x, y \in \mathbb{Q}, x^2 - dy^2 = -1 \right\} = \\ &= T' \cup T' \begin{pmatrix} x_0 & -y_0 d \\ y_0 & -x_0 \end{pmatrix}, \end{aligned}$$

where (x_0, y_0) is a fixed solution of the equation $x^2 - dy^2 = -1$; otherwise $\mathcal{N}_{\text{SL}(2, \mathbb{Q})} T' = T'$. Thus, the lower garland of the lattice $\text{Lat}(T', \text{SL}(2, \mathbb{Q}))$ consists either of one or of two subgroups.

3⁰. Let K/k be a pure inseparable field extension of degree $q = p^n$, where $\text{char } k = p$. Since $N_{K/k}(\alpha) = \alpha^q$, we see that $\text{Ker } N_{K/k} = 1$. Indeed, $\alpha^q = 1$ iff $(\alpha - 1)^q = 0$.

§ 8. AN ELEMENTARY APPROACH

Let K be a finite separable extension of an infinite field k , N a fixed natural number.

Lemma 8.1. *Let $x \in K \setminus k$. Then the set of $\alpha \in k$ such that $(x + \alpha)^N \in k$ is finite, moreover, it can contain not more than N elements.*

Proof. We denote by p the characteristic exponent of k . Then $N = p^r N'$, where $\gcd(N', p) = 1$. In view of the separability of K/k we can assume that $\gcd(N, p) = 1$.

Let's suppose that the assertion of Lemma is false and choose different $\alpha_0, \dots, \alpha_N \in k$ such that $(x + \alpha_i)^N \in k$ for every $i = 0, 1, \dots, N$. Using the standard binomial formula, we get a system of equations of the form $CX = Y$, where $C \in \text{GL}(N + 1, k)$, $Y \in k^{N+1}$, whence $X \in k^{N+1}$, and, in particular, $Nx \in k$, therefore $x \in k$. A contradiction.

Corollary. *There exist infinitely many primitive elements of the extension K/k with the norm 1.*

Proof. Let's take a primitive element x of the finite separable extension K/k . For every $\alpha \in k$ one can consider the element $x_\alpha = \frac{(x+\alpha)^n}{N_{K/k}(x+\alpha)}$, where $n = (K : k)$. All such elements have the norm 1. Since there are only finitely many intermediate subfields F , $k \subseteq F \subseteq K$, it follows from Lemma 8.1 that there are infinitely many different primitive among the elements x_α .

If k is a finite field, then, as it can be easily verified, there exists a primitive element of the extension K/k with the norm 1.

Let $R = k$ be a finite field, $S = K_1 \oplus \dots \oplus K_t$, where K_i/k are finite field extensions of degree n_i , $\tilde{G} = \text{SL}(n, \bar{k})$.

Lemma 8.2. *If $S \neq \mathbb{F}_3 \oplus \mathbb{F}_3$, then $\mathcal{N}_{G'}T' = \mathcal{N}_{\text{GL}(n,k)}T \cap G'$.*

Proof. We show that the set S' which consists of k -linear combinations of the elements $s \in S$ such that $t(s) \in G'$ contains S^* . Then the condition $(*)$ is fulfilled, and we can use Lemma 3.4.

A. Let not all of K_i/k be 1-dimensional. For every index j such that $n_j \geq 2$ we choose a primitive element α_j of the extension K_j/k with the norm 1. Then $(0, \dots, \alpha_j - 1, \dots, 0) \in S'$, and we get $K_j \subseteq S'$.

If some of K_i are 1-dimensional, then there is nothing to prove for $k = \mathbb{F}_2$. If $k \neq \mathbb{F}_2$, then for every j such that $n_j = 1$, one can choose an element $c_j \in K_j = k$ not equal to 0 and 1. There is no loss in assuming $n_1 \geq 2$. In view of the surjectivity of the norm homomorphism one can find an element $\alpha \in K_1$ with the norm c^{-1} . Then it is clear that $(0, \dots, c_j - 1, \dots, 0) \in S'$, therefore $K_j \subseteq S'$.

B. Let all $K_i = k$ and $|k| \geq 4$. Let's choose $c \in k^*$ such that $c^2 \neq 1$. Then $(c^2 - 1, c - 1, \dots, c - 1, 0) \in S'$, whence $(c + 1, 1, \dots, 1, 0) \in S'$. If we act in the same manner with $c + 1 \neq 0$, we get $(1, 0, \dots, 0) \in S'$.

C. It remains to examine the case $K_i = \mathbb{F}_3$, $t \geq 3$. If t is odd, then $(1, -1, \dots, -1) + (1, \dots, 1) = (-1, 0, \dots, 0) \in S'$. If t is even, then by the same reasons $(-1, -1, 0, \dots, 0) \in S'$, and also $(-1, 1, 0, \dots, 0) = (1, -1, 1, -1, \dots, -1) + (1, -1, -1, 1, \dots, 1) \in S'$. Lemma is proved.

Remark. It's easy to verify that for $S = \mathbb{F}_3 \oplus \mathbb{F}_3$ the assertion of Lemma 8.2 doesn't hold true.

Corollary. *Let $R = k$ be a field, $S = K_1 \oplus \dots \oplus K_t$, where K_i/k are finite separable field extensions, and:*

- (i) *not more than two of them are equal to \mathbb{F}_2 ;*
- (ii) *$S \neq \mathbb{F}_3 \oplus \mathbb{F}_3$.*

Then the normalizer of T' in G' is equal to the intersection of the semidirect product of T and $\text{Aut}(S/R)$ with the group G' .

G.Seitz [S] obtained an almost exhaustive description of the intermediate subgroups in the finite groups of Lie type which contain a group of rational points of

a maximal torus in the corresponding algebraic group. For simplicity we state his results only for Chevalley groups.

Let $\tilde{G} = G(\Phi, \bar{k})$ be a Chevalley group of type Φ over the algebraic closure \bar{k} of a finite field $k = \mathbb{F}_q$, where $\text{char } k \neq 2, 3$ and $q \geq 13$, $G = G(\Phi, k)$. It was proved in [S] that the group of k -rational points T of an arbitrary maximal torus \tilde{T} in \tilde{G} is a parnormal subgroup of G (see [BB], [V2]), which means, in particular, that the lower garland of the lattice of intermediate subgroups of G containing T coincides with the interval $(T, \mathcal{N}_G T)$.

Therefore one can elaborate the statements of the corresponding corollaries from Theorems 6.2 and 7.3.

For those fields k which do not satisfy the conditions under which the results of Seitz hold true, the lower garland of the lattice $\text{Lat}(T, G)$ actually may not coincide with the interval $\text{Lat}(T, \mathcal{N}_G T)$. The examples of such kind can be easily constructed for $k = \mathbb{F}_3$, $K = (\mathbb{F}_3)^n$, $n \geq 2$.

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